Adelic Poisson Summation Formula

Pranav Setpal

Purdue University

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Outline

Poisson Summation Formula for $\mathbb{R}$

$$\sum_{k\in\mathbb{Z}} f(k) = \sum_{k\in\mathbb{Z}} \hat{f}(k)$$

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Generalizng Fourier Transform to LCA Groups G

- Fourier Transform for $\mathbb{R}$: $\hat{f}(\xi) = \int_{\mathbb{R}} f(x)\, e^{-2\pi i \xi x} \, \mathrm{d}x$

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- Fourier Transform for G: $\hat{f}(\chi) = \int_G f(x) \overline{\chi(x)} \, \mathrm{d}x$

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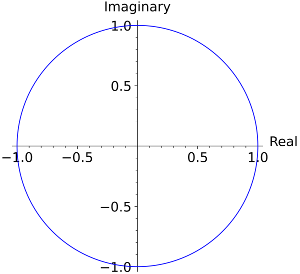
- Fourier Transform for G: $\hat{f}(\chi) = \int_G f(x) \overline{\chi(x)} \, \mathrm{d}x$

We take G to be locally compact and abelian/commutative (LCA)

Why do we take conjugate? Just convention.

The Dual of G

$\chi \in \{$group homomorphisms from G to the complex unit circle$\}$



We call this the dual of G
$\widehat{G} = \{\chi_1, \chi_2, \cdots\}$

$\widehat{G}$ is currently an abelian group.

Definition (Group Operation on $\widehat{G}$)

$$\chi(x \cdot y) := \chi(x) \cdot \chi(y)$$

If we endow $\widehat{G}$ with the *compact-open* topology,
it becomes locally compact.

Definition (Compact-Open Topology)

For $K \subseteq G$ compact and $U \subseteq \mathbb{C}$ open, define open sets as

$$L(K, U) = \{\chi \in \widehat{G} : \chi(K) \subseteq U\}$$

Hence, we can define $\hat{\hat{f}} : \widehat{\widehat{G}} \to \mathbb{C}$, the Fourier Transform of $\hat{f}$

Outline

We can easily relate G and $\widehat{\widehat{G}}$ with the continuous group homomorphism

$$\delta : x \mapsto \delta_x \quad \text{where} \quad \delta_x(\chi) = \chi(x)$$

How to tame Dual-ception

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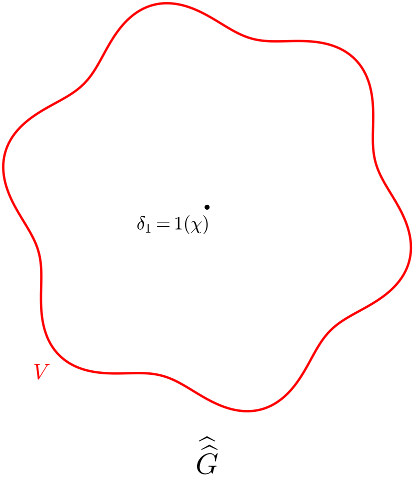
Theorem (Pontryagin Duality)

$G \cong \widehat{\widehat{G}}$ algebraicly and topologically (everything we care about) by the map δ defined above

Proof for Continuity of δ



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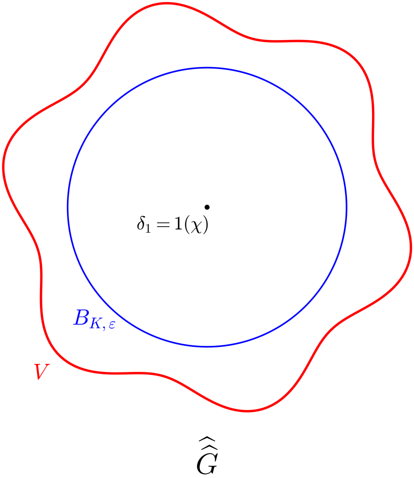
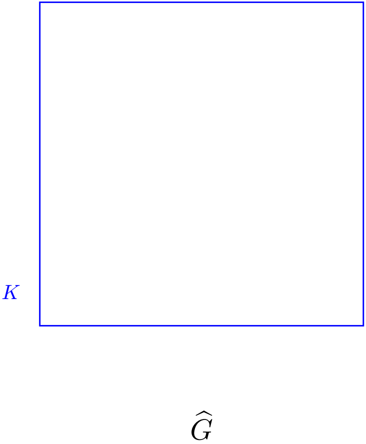


Proof for Continuity of δ (cont.)



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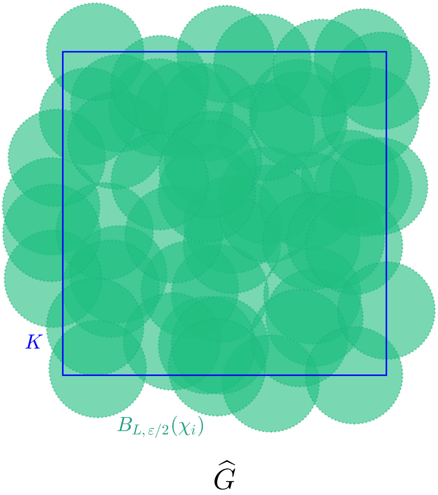

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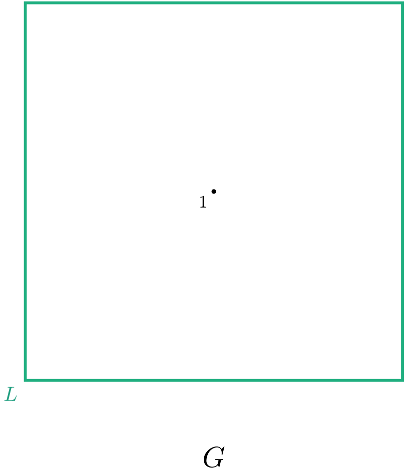

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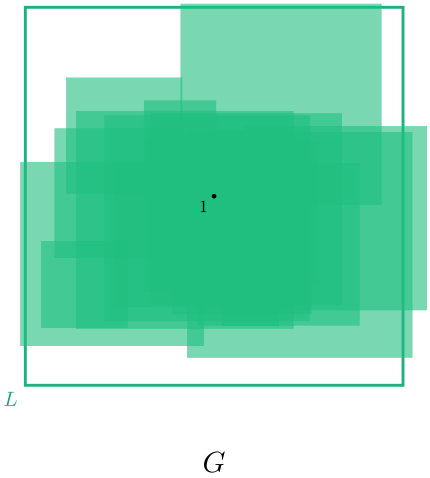

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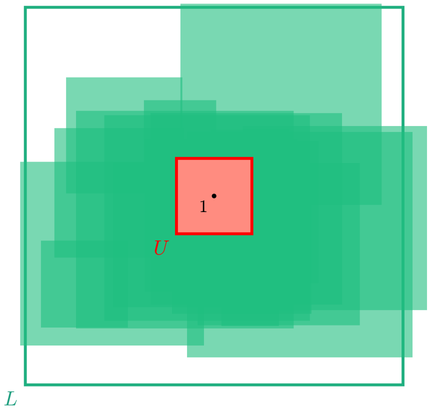

G

Using Pontryagin Duality, we get the Fourier Inversion Formula

$$f(x) = \hat{\hat{f}}(\delta_{x^{-1}}) = \int_{\widehat{G}} \hat{f}(\chi)\chi(x)\,\mathrm{d}\chi$$

For the Fourier Transform on $\mathbb{R}$, you may have seen this as

$$f(x) = \int_{\mathbb{R}} \hat{f}(\xi)e^{2\pi i\xi x}\,\mathrm{d}\xi$$

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One small thing before we continue

We will be taking quotients ${}^{G}\!\big/_{H}$ of our locally compact and abelian group G, so let's note some important statements about it.

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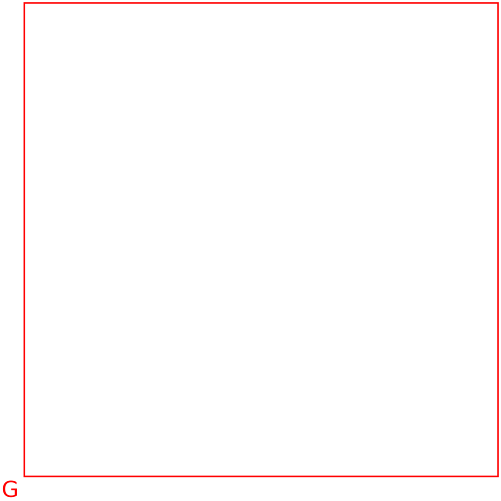
- It is abelian.
- We can endow it with a locally compact topology.
- For any integrable function f,

$$\int_G f(x)\,\mathrm{d}x = \int_{{}^{G}\!/_{H}} \int_H f(xh)\,\mathrm{d}h\,\mathrm{d}(xH)$$

Define $f^H(x) = \int_H f(xh)\,\mathrm{d}h$ for convenience

Intuition for Iterated Integral



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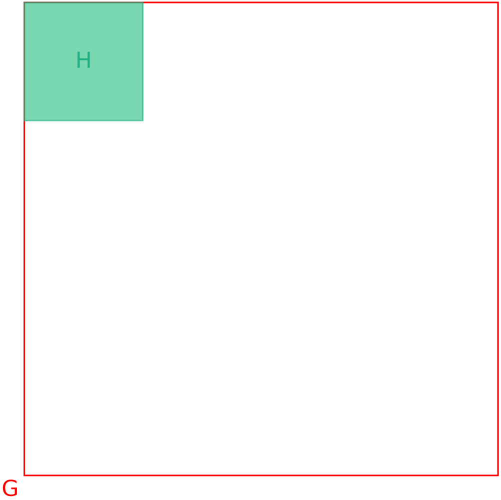

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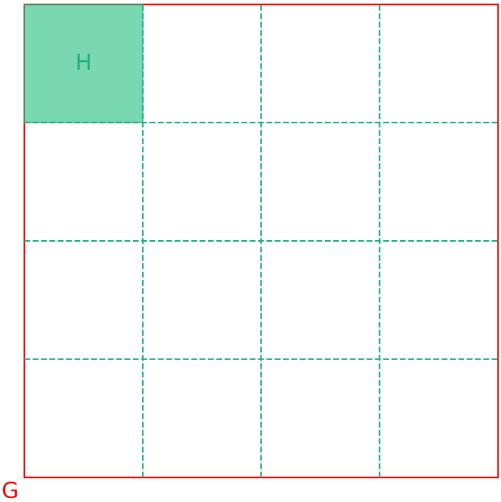

Short exact sequences

Let H be a subgroup of G
We have the below short sequence
(so τ injective and q surjective with H mapped to identity)

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Hence, we get the short sequence

$$1 \longrightarrow \widehat{{}^{G}\!/_{H}} \xrightarrow{q'} \widehat{G} \xrightarrow{\tau'} \widehat{H} \to 1$$

Short exact sequence for $\mathbb{R}$

For $G = \mathbb{R}$ and $H = \mathbb{Z}$, we have the short sequences

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It turns out that $\mathbb{R} \cong \widehat{\mathbb{R}}$ by $i : \xi \mapsto \chi_\xi$ where $\chi_\xi(x) = e^{2\pi i \xi x}$
In fact, we identify them using a non-trivial character
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Using this, we can infer $\widehat{\mathbb{R}/\mathbb{Z}} \cong \mathbb{Z}$ and $\widehat{\mathbb{Z}} \cong \mathbb{R}/\mathbb{Z}$

Theorem (Poisson Summation Formula)

$$\int_H f(xh)\,\mathrm{d}h = \int_{\widehat{G/H}} \hat{f}(\chi)\chi(x)\,\mathrm{d}\chi$$

Proof.

For $\chi \in \widehat{G/H}$,
$$\begin{aligned}\widehat{f^H}(\chi) &= \int_{G/H} f^H(xH)\overline{\chi(xh)}\,\mathrm{d}(xH) \\ &= \int_{G/H}\int_H f(xh)\overline{\chi(x)}\,\mathrm{d}h\,\mathrm{d}(xH) \\ &= \int_G f(x)\overline{\chi(x)}\,\mathrm{d}x\end{aligned}$$

Thus, $\widehat{f^H}(\chi) = \hat{f}(\chi)$ for $\chi \in \widehat{G/H} \iff \widehat{f^H}(\chi) = \hat{f}(\chi)|_{\widehat{G/H}}$

Poisson Summation Formula (cont.)

Proof (cont.)

$$
\begin{aligned}
\int_H f(xh)\,\mathrm{d}h &= f^H(x) \\
&= \widehat{\widehat{f^H}}(\delta_{x^{-1}}) \\
&= \widehat{\hat{f}\big|_{\widehat{G/H}}}(\delta_{x^{-1}}) \\
&= \widehat{\hat{f}\big|_{\widehat{G/H}}}(\overline{\chi})
\end{aligned}
$$

$$
\therefore \int_H f(xh)\,\mathrm{d}h = \int_{\widehat{G/H}} \hat{f}(\chi)\chi(x)\,\mathrm{d}\chi
$$

$\square$

Poisson Summation Formula: Special Case

Theorem (Poisson Summation Formula)

$$\int_H f(xh)\,\mathrm{d}h = \int_{\widehat{G/H}} \hat{f}(\chi)\chi(x)\,\mathrm{d}\chi$$

Taking $x = 1$, we get

$$\int_H f(h)\,\mathrm{d}h = \int_{\widehat{G/H}} \hat{f}(\chi)\,\mathrm{d}\chi$$

What we have:

- $\mathbb{R} \cong \widehat{\mathbb{R}}$
- $\widehat{\mathbb{R}/\mathbb{Z}} \cong \mathbb{Z}$
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Behold

$$\sum_{k\in\mathbb{Z}} f(k) = \sum_{k\in\mathbb{Z}} \hat{f}(k)$$

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$\mathbb{R}$ is not special

In math, we have no freedom in defining the rational numbers $\mathbb{Q}$. It is the smallest infinite set containing the integers in which you can add, subtract, multiply and divide.

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To define $\mathbb{R}$ however, we *choose* a distance function, namely $|\cdot|$, and include all the limit points found.

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Theorem (Ostrowski's Theorem)

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By wishful thinking,

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By wishful thinking,

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In truth, the best we can have is

$$\mathbb{A} = \bigcup_{\text{finite } P \subseteq \text{ all primes}} \left(\mathbb{R} \times \prod_{p \in P} \mathbb{Q}_p \times \prod_{p \notin P} \mathbb{Z}_p \right)$$

Outline

Short exact sequence for $\mathbb{A}$

For $G = \mathbb{A}$ and $H = \mathbb{Q}$, we have the short sequences

$$1 \longrightarrow \mathbb{Q} \xrightarrow{\tau} \mathbb{A} \xrightarrow{q} {}^{\mathbb{A}}\!/_{\mathbb{Q}} \to 1$$

and

$$1 \longrightarrow \widehat{{}^{\mathbb{A}}\!/_{\mathbb{Q}}} \xrightarrow{q'} \widehat{\mathbb{A}} \xrightarrow{\tau'} \widehat{\mathbb{Q}} \to 1$$

It turns out that $\mathbb{A} \cong \widehat{\mathbb{A}}$, by identifying them using a nowhere trivial character $\chi_\xi(x) = \chi_1(\xi x)$

Using this, we can infer $\widehat{{}^{\mathbb{A}}\!/_{\mathbb{Q}}} \cong \mathbb{Q}$ and $\widehat{\mathbb{Q}} \cong {}^{\mathbb{A}}\!/_{\mathbb{Q}}$

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Setting $G = \mathbb{A}$ and $H = \mathbb{Q}$, we get $\displaystyle\sum_{k\in\mathbb{Q}} f(k) = \sum_{\chi\in\widehat{\mathbb{A}/\mathbb{Q}}} \hat{f}(\chi)$

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Now that we know $\mathbb{A} \cong \widehat{\mathbb{A}}$, let's redefine $\widehat{\mathbb{A}}$ so that $\mathbb{A} = \widehat{\mathbb{A}}$.
Behold

$$\sum_{k\in\mathbb{Q}} f(k) = \sum_{k\in\mathbb{Q}} \hat{f}(k)$$

Thank you

Note: I have made no contributions here!

Generously used facts from:

- *Principles of Harmonic Analysis, 2nd Edition*, by *Anton Deitmar* and *Siegfried Echterhoff*
- *Algebraic Number Theory* by *Jürgen Neukirch*

Thank you to Dr. Daniel Johnstone for his guidance and mentorship through learning all this.

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Thank you to everyone here listening to me ramble.